

A Study on Convergence Analysis of Adomian Decomposition Method Applied to Different linear and non-linear Equations

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Abstract: The convergence of any method is too important to mobilize it. In this work, the analysis of convergence of the Adomian decomposition method (ADM) is confabulated for different linear and non-linear equations i.e. integral equations, differential equations, and initial value problems. The order and accuracy of ADM are a contest. In the applications of ADM, the method is exploited to solve the Heat and Wave equation to authenticate the accuracy and rapid convergence of ADM.

Keywords: Adomian decomposition method, Convergence analysis of ADM, ADM applied to linear and no-linear equations, Applications of ADM, ADM applied to wave equation.



1. Introduction:

The Adomian decomposition method (ADM) was firstly introduced by the American physicist G-Adomian (1923-1996) at the start of the 1980s [1, 2, 3, 4]. Then Adomian and others have successfully applied the decomposition technique for many non-linear equations such as polynomial, Trigonometric, Composite, hyperbolic, exponential, radial, and decimal power non-linear equations [5, 6, 7]. The main procedure of this technique is to decompose the solution into series $u = \sum_{n=0}^{\infty} u_n$ such that every next term can find by recurrence relation[8].

In many papers Adomian and Bellman [1], [9, 10, 11, 12], Adomian [13] has proposed the decomposition technique to solve linear and non-linear deterministic equations, Ordinary differential equations, partial differential equation, Integral equations, and initial value problems. The ADM method is a uniquely powerful technique that consigns the analytic and numerical solutions of the differential equations[9]. The solution obtained by Adomian decomposition is pre-existent, Analytic, Unique, and satisfy the problem, Which is under consideration [9].To solve non-linear differential equations decomposition method is different from other methods such as successive approximations, Sinc-Galerkin, finite difference, ad hoc transformation, or perturbation methods [1], [11], [12], [14]. This method avoids a long integration of the Picard method and solves the problem more efficiently which can not solve by other methods(Iterative method)[15]. In non-linear equations, the non-linear term can be solved by a well known polynomial Adomian's polynomial[16]. This method is used especially for biomedical problems leading to mathematical modeling [15], [17]. Tonningen commences that the ADM method can be used to solve engineering problems and accommodate a paramount and convergent solution without linearizing or considerateness[18].

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However, without the proof of convergence, any method has low significance to use especially for non-linear solutions (Because non-linear may have singularities) [19]. Firstly Cherruault counsel some necessary conditions and techniques to determine the convergence of ADM who used the fixed point theorem for functional equations [20]. And theoretical treatment of convergence is deliberated by Cherruault in [15], [21], [22], [23]. Also, other researchers expose frequent papers on the convergence of ADM in [15, 23, 24, 25, 26]. Then Gabet also proposed the new definition for the proof of convergence of ADM [27, 28, 29]. Cherruault and Abbaoui proposed a new formula for Adomian's polynomial [30].

Applications of distinctive types of integral equations by using ADM is deliberated in many papers [31], [32], [33], [34]. However to treasure the clarification of Fredholm integral equation numerous papers were ventilated with different methods and technique i.e. Chebyshev method [35], [36], Taylor series method [37], block-pulse function method [38], hat-basis functions method [39], homotopy method [40, 41] and Taw method [42]. But the ADM method is the preeminent method to interpret the second kind of voletra integrodifferential equation [43], [32].

Recently, the ADM method has been applied to treasure the approximate and correct resolution of differential equations [44]. Many papers published to find out the formal and accessible solution of partial differential equations [45], [46], [47]. In [1] Adomian discussed the convergence of ADM for differential and integral equations by using a fixed point theorem. Eugene proved the convergence of ADM by applying this method on the reaction-convection-diffusion equation which delineates the dissipation of chemically reactive materials in [48].

The convergence of ADM for the initial value problem is to consult within [49]. In [50] the comparison of quintic C^2 -spline integration methods [51] with decomposition method [1], [3] is bounce off for initial value problems. Then large number of papers were broadcasted to dig out the comparative solution of initial value problems [52], [53], [54], [55].

The order of convergence of ADM is confabulated by Biazar and Babolian in [56]. Gorden and Boumenir familiarize the Rate of convergence of ADM [19]. El-Kalla kicks off another view for error analysis of ADM [57].

The augmentation of total work can be encapsulated in the following EIGHT points.

- Introducing the convergence of ADM.
- The convergence of ADM applied to Integral equation i.e. Integrodifferential equation
- The convergence of ADM applied to the Differential and functional equation.
- The convergence of ADM applied to the initial value problem.
- Order of convergence of ADM.
- New Ideas for the convergence of ADM
- New modifications for Adomian polynomials
- Applications of ADM applied to heat and wave equation.

2. Explanation:

2.1 Convergence of ADM applied to an integral equation

In [58] the decomposition method is utilized to solve a non-linear Fredholm integral equation. Different modeling non-linear problems are solved by ADM in [59]. To convince the convergence of ADM supposes the 2nd kind of non-linear Fredholm integral equation.

$$u(t) = x(t) + \int_0^t k(t, \beta) f(u(\beta)) d\beta \quad (1)$$

Suppose $x(t)$ is bounded $\forall t \in [0, T]$ and $|k(t, \beta)| \leq M$. Where $f(u)$ is non-linear and it is Lipchitz continuous and has the Adomian polynomial representation such that

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n) \quad \text{with}$$

$$A_n = \left(\frac{1}{n!}\right) \frac{d^n}{d\lambda^n} \left[f\left(\sum_{j=0}^{\infty} \lambda^j u_j\right) \right]_{\lambda=0}$$

In [60] another formula for Adomian polynomial is deduced such that

$$A_n = f(S_n) - \sum_{j=0}^{n-1} A_j \tag{2}$$

And the Adomian series for the function $u(t)$ is

$$u(t) = \sum_{j=0}^{\infty} u_j(t) \tag{3}$$

Now we will prove firstly convergence theorem for integral equation and then its uniqueness [61].

Theorem 2.1.1 (Convergence theorem). The series solution $u(t) = \sum_{j=0}^{\infty} u_j(t)$ of non-linear Fredholm integrodifferential equation (1) By ADM will converge if $0 < \alpha < 1$ and $|u_1| < \infty$.

Proof: Consider a Banach space $(C[a, b], d)$ of all consecutive functions denominated on interval h . And suppose $\{S_n\}$ be a sequence with an arbitrary partial sum S_p and S_q , $p \geq q$. We have to show that the sequence $\{S_p\}$ is a Cauchy disposition in this Banach space.

$$\begin{aligned} \|S_p - S_q\| &= \max_{t \in h} |S_p - S_q| \\ &= \max_{t \in h} \left| \sum_{j=m+1}^p u_j(t) \right| \\ &= \max_{t \in h} \left| \sum_{j=q+1}^p \int_0^t k(t, \beta) A_{j-1} d\beta \right| \\ &= \max_{t \in h} \left| \int_0^t k(t, \beta) \sum_{j=q}^{p-1} A_j d\beta \right| \end{aligned}$$

By using the formula of Adomian polynomial (2), then we get

$$\begin{aligned} \|S_p - S_q\| &= \max_{t \in h} \left| \int_0^t k(t, \beta) [f(S_{p-1}) - f(S_{q-1})] d\beta \right| \\ &\leq \max_{t \in h} \int_0^t |k(t, \beta)| |f(S_{p-1}) - f(S_{q-1})| d\beta \\ &\leq \alpha \|S_{p-1} - S_{q-1}\| \end{aligned}$$

Now Suppose $p = q + 1$; then

$$\begin{aligned} \|S_{q+1} - S_q\| &\leq \alpha \|S_q - S_{q-1}\| \leq \alpha^2 \|S_{q-1} - S_{q-2}\| \\ &\leq \dots \leq \alpha^q \|S_1 - S_0\| \end{aligned}$$

By using the triangular inequality we have that

$$\begin{aligned} \|S_p - S_q\| &\leq \|S_{p+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \dots \\ &\quad + \|S_p - S_{p-1}\| \\ &\leq [\alpha^q + \alpha^{q+1} + \alpha^{q+2} + \dots + \alpha^{p-1}] \|S_1 - S_0\| \\ &\leq \alpha^q [1 + \alpha + \alpha^2 + \dots + \alpha^{p-q-1}] \|S_1 - S_0\| \\ &\leq \alpha^q \left(\frac{1 - \alpha^{p-q}}{1 - \alpha}\right) \|u_1(t)\| \end{aligned}$$

As $0 < \alpha < 1$ then we have $(1 - \alpha^{p-q}) < 1$; then

$$\|S_p - S_q\| \leq \frac{\alpha^q}{1 - \alpha} \max_{t \in h} |u_1(t)| \tag{4}$$

But as $|u_1|$ is bounded so $\|S_p - S_q\| \rightarrow 0$ as $q \rightarrow \infty$. Which proves that the sequence $\{S_p\}$ is a Cauchy sequence and the convergence of this theorem. Now to prove the uniqueness

Theorem 2.1.2 (Uniqueness theorem). The above equation (1) has a unique solution if $0 < \alpha < 1$, $\alpha = LMT$.

Proof: Let u and u^* be two different solutions then

$$\begin{aligned} |u - u^*| &= \left| \int_0^t k(t, \beta) [f(y) - f^*(y)] d\beta \right| \\ &\leq \int_0^t |k(t, \beta)| |f(y) - f^*(y)| d\beta \end{aligned}$$

$$\leq LM|u - u^*| \int_0^t d\beta$$

$$\leq \alpha|u - u^*|$$

From this we have $(1 - \alpha)|u - u^*| \leq 0$ as $0 < \alpha < 1$ then $u - u^* = 0 \Rightarrow u = u^*$ which shows the uniqueness of this theorem.

Theorem 2.1.3 (Absolute truncation error estimate): The maximum absolute truncation error of the series $u(t) = \sum_{j=0}^{\infty} u_j(t)$ to equation (1) is estimated as

$$\max_{\forall t \in h} \left| u(t) - \sum_{j=0}^q u_j(t) \right| \leq \frac{K\alpha^{q+1}}{L(1-\alpha)} \text{ where } K = \max_{\forall t \in h} |f(x(t))|$$

Proof: From the above convergence theorem we have that, from equation (4)

$$\|S_p - S_q\| \leq \frac{\alpha^q}{1-\alpha} \max_{\forall t \in h} |u_1(t)|$$

When $p \rightarrow \infty$ then $S_p \rightarrow u(t)$.so

$$\|u(t) - S_q\| \leq \frac{\alpha^{q+1}}{L(1-\alpha)} \max_{\forall t \in h} |f(x(t))|$$

So the maximum absolute truncation error will be

$$\max_{\forall t \in h} \left| u(t) - \sum_{j=0}^q u_j(t) \right| \leq \frac{K \alpha^{q+1}}{L(1-\alpha)}$$

Which completes the proof [61, 62, 63].

Now from the applications of non-linear integral equations, we apply the convergence of ADM to non-linear Volterra integrodifferential equation.

2.1.4 The convergence of ADM applied to non-linear Volterra integrodifferential equations.

A comparison between Wavelet-Galerkin and ADM technique is discussed in [64] for integrodifferential equation and concluded that ADM gives a more accurate and streamline solution than the Wavelet method. In this work, a type of non-linear integral equation i.e non-linear integrodifferential equation will be discussed to prove the convergence,

uniqueness, and error analysis of ADM [60]. So take a non-linear integrodifferential equation such that

$$\frac{d^k}{dt^k} u(t) = x(t) + \int_a^t k(t, \beta) f(u(\beta)) d\beta \quad (5)$$

With initial conditions

$$u(a) = c_0, \quad u'(a) = c_1, \\ u''(a) = c_2 \quad \dots \quad u^{k-1}(a) = c_{k-1}$$

Where $x(t)$ is bounded $\forall t \in h = [a, b]$ and the non-linear term $f(u(x))$ assumed to be analytic and has the Adomian's polynomial representation such that

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n) \text{ with} \\ A_n = \left(\frac{1}{n!}\right) \frac{d^n}{d\lambda^n} \left[f\left(\sum_{n=0}^{\infty} \lambda^j u_j\right) \right]_{\lambda=0} \quad (6)$$

and the solution $u(x)$ in the series form will be $u(t) = \sum_{n=0}^{\infty} u_n(t)$. Now for the convergence and uniqueness consider the following theorem [61, 63]

Theorem 2.1.5 (Convergence). The solution of the non-linear integrodifferential equation (5) will converges by ADM if $f(u)$ satisfies the Lipchitz conditions in the interval h and has the unique solution if $MM_1 < \frac{(t-a)^{k+1}}{(t+1)!} < 1$, Where M is a Lipchitz constant.

Proof: Suppose a complete metric space $(C[a, b], d)$ on interval h with distance formula.

$$d(f_1(t), f_2(t)) = \max_{\forall t \in h} |f_1(t) - f_2(t)|$$

Suppose the sequence S_n , $S_n = \sum_{j=0}^n u_j(t) = u_0 + u_1 + u_2 + \dots + u_j$ then

$$f(S_n) = f\left(\sum_{j=0}^{\infty} u_j(t)\right) \\ = \sum_{j=0}^{\infty} A_j(u_0 + u_1 + u_2 + \dots + u_j)$$

Suppose S_n, S_m arbitrary partial sum, $\geq m$, We have to prove that the above sequence is a Cauchy sequence.

$$\begin{aligned}
 d(S_n, S_m) &= \max_{t \in h} |S_n - S_m| \\
 &= \max_{t \in h} \left| \sum_{j=m+1}^n u_j(t) \right| \\
 &= \max_{t \in h} \left| \sum_{j=m+1}^{\infty} L^{-1} \int_a^t k(t, \beta) A_{j-1} d\beta \right| \\
 &= \max_{t \in h} \left| L^{-1} \int_a^t k(t, \beta) \sum_{j=m}^{n-1} A_j d\beta \right| \\
 &= \max_{t \in h} \left| L^{-1} \int_a^t k(t, \beta) [f(S_{n-1}) - f(S_{m-1})] d\beta \right| \\
 &\leq \max_{t \in h} L^{-1} \int_a^t |k(t, \beta)| | [f(S_{n-1}) - f(S_{m-1})] | d\beta \\
 &\leq M \max_{t \in h} | [f(S_{n-1}) - f(S_{m-1})] | L^{-1} \int_a^t d\beta
 \end{aligned}$$

As $f(u)$ is Lipchitz so

$$\begin{aligned}
 d(S_m, S_n) &\leq MM_1 \max_{t \in h} |S_{m-1} - S_{n-1}| \int_a^t \dots (k \\
 &\quad + 1) - fold \dots \int_a^t d\beta \dots d\beta \\
 &\leq MM_1 \frac{(t-a)^{k+1}}{(t+1)!} d(S_{m-1}, S_{n-1}) \\
 &\leq \alpha d(S_{m-1}, S_{n-1}) \quad \text{And} \\
 \alpha &= MM_1 \frac{(t-a)^{k+1}}{(t+1)!}
 \end{aligned}$$

Now for $= m + 1$, Then

$$\begin{aligned}
 d(S_{m+1}, S_m) &\leq \alpha d(S_m, S_{m-1}) \\
 &\leq \alpha^2 d(S_{m-1}, S_{m-2}) \\
 &\leq \dots \leq \alpha^m d(S_1, S_0)
 \end{aligned}$$

By helping the triangular inequality

$$\begin{aligned}
 d(S_m, S_n) &\leq \alpha [d(S_{m-1}, S_m) + d(S_m, S_{m+1}) \\
 &\quad + \dots d(S_{n-2}, S_{n-1})] \\
 &\leq \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha} d(S_1 - S_0) \\
 &\leq \frac{\alpha^m}{1 - \alpha} d(S_1, S_0)
 \end{aligned}$$

But $d(S_1, S_0) = \max_{t \in h} |S_1 - S_0| = \max_{t \in h} |u_1|$ is bounded, so the series $\sum_{n=0}^{\infty} u_n(t)$ converges.

Uniqueness: To prove the uniqueness suppose that u and u^* are two different solutions.

$$\begin{aligned}
 d(u, u^*) &= \max_{t \in h} \left| L^{-1} \int_a^t k(t, \beta) [f(u) - f^*(u)] d\beta \right| \\
 &\leq \max_{t \in h} L^{-1} \int_a^t |k(t, \beta)| |f(u) - f^*(u)| d\beta \\
 &\leq M_1 \max_{t \in h} |f(u) - f^*(u)| L^{-1} \int_a^t d\beta \\
 &\leq \alpha d(u, u^*)
 \end{aligned}$$

$(1 - \alpha) d(u, u^*) \leq 0$ and $0 < \alpha < 1$ then $d(u, u^*) = 0$ so $u = u^*$ which shows the uniqueness of the above theorem [61, 63]. So the series obtained from this method will be convergent absolutely and uniformly [25]. This method reduces the mathematical work and also computational work [65].

2.2 Convergence of ADM utilizing the differential equation

Initially, it was the major problem to solve different types of the differential equation such as Cauchy problem. Different methods were applied such as the Taylor collection method [66], differential transform method [67], homotopy perturbation method [68], variational iteration method [69], and homotopy-analysis method [70] but Adomian decomposition method is best for rapidly convergent series. Eugene used the decomposition method on the reaction-convection-diffusion

equation which is used to constitute of dissipation of reactive material to prove the convergence [48]. An 8th order differential equation used intensional vibration of uniform beam is discussed in [71]. So for the convergence of ADM suppose a differential equation such that

$$\frac{dy}{dx} = f(y) + g \tag{7}$$

With $y(x)|_{x=0} = c_0$

The Adomian series solution in the series form is such that

$$y_n = \sum_{n=0}^{\infty} y_n \tag{8}$$

And the non-linear term

$$f(y) = \sum_{n=0}^{\infty} A_n \tag{9}$$

Where A_n 's are the Adomian polynomials and these polynomials can be determined such that

$$z = \sum_{j=0}^{\infty} \lambda^j y_j, \quad f\left(\sum_j \lambda^j y_j\right) = \sum_{j=0}^{\infty} \lambda^j A_j$$

The Adomian's polynomials can also be determined by the formula[12]

$$A_n = \left(\frac{1}{n!}\right) \frac{d^n}{d\lambda^n} f\left(\sum_j \lambda^j y_j\right) \Bigg|_{\lambda=0}$$

Putting equation (8) and (9) in equation (7) we get

$$\sum_n y_n = c_0 + L^{-1} g(x) + L^{-1} \sum_n A_n$$

Now every term of the series (8) can be calculated as

$$y_0 = c_0 + L^{-1} g$$

$$y_1 = L^{-1} A_0$$

⋮

$$y_{n+1} = L^{-1} A_n$$

Then from this Adomian's series the approximate solution of the series $\sum_{n=0}^{\infty} y_n$ can be determined such that.[30]

$$\phi_n = \sum_{j=0}^{j-1} y_j \quad \text{with} \quad \lim \phi_n = y$$

In same manners, ADM can also be applied on functional equations such that

2.2.1 ADM applied to a functional equation

Adomian and Bougoffa accomplish some results of the speed of convergence of ADM to solve linear and non-linear functional equations [21], [72]. Consider a functional equation for the basic implementation of the ADM method

$$y - N(y) = f \tag{10}$$

Where f is given function and N is a non-linear operator from Hilbert Space $H \rightarrow H$ then by Adomian series[73] the function y can be represented in series, such that

$$y = \sum_{j=0}^{\infty} y_j \tag{11}$$

And the non-linear term can be represented as

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{12}$$

Where A_n 's are the Adomian Polynomials then

$$z = \sum \lambda^j y_j, \quad N\left(\sum \lambda^j y_j\right) = \sum_{n=0}^{\infty} \lambda^n A_n$$

And A_n can be derived as

$$n! A_n = \frac{d^n}{d\lambda^n} \left[N\left(\sum \lambda^j y_j\right) \right] \tag{13}$$

And $n = 0, 1, 2, 3, 4, \dots$

And the Adomian series will be as

$$S_n = y_1 + y_2 + y_3 + \dots + y_n \tag{14}$$

Or by the Iterative method

$$S_{n+1} = N(y_0 + S_n) \quad , S_0 = 0 \quad (15)$$

Confederated with the functional equation

$$S = N(y_0 + S) \quad (16)$$

Cherruault used a fixed pint theorem for the numerical solution of equation (10) [74], [75]. And in [20] Cherruault defines the convergence theorem such that.

2.2.2 Theorem: (Convergence theorem).

Suppose N be the contraction such that $(\delta < 1)$, and if we assume that $\|N_n - N\| = \epsilon_n \rightarrow_{n \rightarrow \infty} 0$ Then the sequence $\{S_n\}$ can be defined as

$$S_{n+1} = N(y_0 + S_n) \quad , S_0 = 0$$

Converges to the solution $N(y_0 + S) = S$.

Proof:

$$\begin{aligned} \|S_{n+1} - S\| &= \|N_n(y_0 + S_n) - N(y_0 + S)\| \\ &= \|N_n(y_0 + S_n) - N(y_0 + S_n) \\ &\quad + N(y_0 + S_n) - N(y_0 + S)\| \\ &\leq \|N_n - N\| \cdot \|y_0 + S_n\| + \delta \|S_n - S\| \\ &\leq \epsilon_n (\|y_0\| + \|S_n\|) + \delta \|S_n - S\| \end{aligned}$$

As $\delta < 1$ in contraction mapping N. And suppose $\|S\| \leq \frac{N_0}{2}$ and $y_0 \leq N_0$. then by recurrence relation $\|S_n - S\| \leq \frac{N_0}{2}$ which involves $\|S_n\| \leq N_0$

$$\|S_{n+1} - S\| \leq \delta \frac{N_0}{2} + 4 \frac{\epsilon_n N_0}{2}$$

If $\|S_{n+1} - S\| \leq (\delta + \epsilon) \frac{N_0}{2}$ then we have to choose $n \geq N$ such that $\|N_n - N\| = \epsilon_n \leq \frac{\epsilon}{4}$. Thus recurrent theorem justified and our proof has been proved [15].

2.3 The convergence of ADM applied to the initial value problem

Initially distinctive Global methods were explored to handle initial value problems [76]. In many papers, the numerical and analytical solution like ADM of initial value problem for the differential equation is discussed such as the power series

solution[77], Rangué Kutta method for initial value problem[78], [79] and Picard mechanism of successive iteration method[80]. The ADM method is also discussed for boundary value problems to partial differential equations[81], [82], [83], [84] and differential equations[85], [86]. Baldwin solves the boundary value problems of six orders used a global-phase integral method in [87].

Convergence Analysis.

Suppose an initial value problem such that

$$\begin{cases} u' = Lu + N(u) \quad , \quad t > 0 \\ u(0) = f \end{cases}$$

Where $L: X \rightarrow Y$ is a linear operator, $N(u)$ is a non-linear function and dashes represent the derivatives. By using Duhamel's principle [88] initial value problem can be written in the integral form such that

$$u(t) = E(t)f + \int_0^t E(t-x)dx \quad (17)$$

Now to prove the convergence and uniqueness suppose an assumption.

Assumption 1. Suppose $L: X \rightarrow Y$ shows the continuous semigroup $E(t) = e^{tL}$ and $C > 0$ is a constant such that

$$\|E(t)f\|_x \leq C \|f\|_x \quad \forall f \in X \quad , \quad \forall t \in \mathbb{R}_+$$

Suppose the non-linear function $N(u)$ be analytic near $u = f$ and X is a Banach algebra with property

$$\|fg\|_x \leq \|f\|_x \|g\|_x \quad \forall f, g \in X$$

Theorem 2.3.1 (Uniqueness theorem)

Suppose $L: X \rightarrow Y$ and the non-linear function $N(u)$ holds the assumption 1. Then there exists a $T > 0$ and a unique solution of initial value problem such that $u(t) \in C([0, T], X) \cap C^1([0, T], Y)$ and $u(0) = f$ and the solution of $u(t)$ depends on initial data f.

Proof: To prove the uniqueness use successive iteration with the free solution $u^{(0)} = E(t)f$ then by the recurrence relation

$$u^{(n+1)}(t) = u^{(0)}(t) + \int_0^t E(t-x)N(u^{(n)}(x))dx \quad (18)$$

For any $\delta > 0$ there exists a $T > 0$, then by using the induction method

$$\begin{aligned} \sup_{t \in [0, T]} \|u^{(n+1)}(t) - f\|_X &\leq \sup_{t \in [0, T]} \|u^{(n+1)}(t) - u^{(0)}(t)\|_X \\ &+ \sup_{t \in [0, T]} \|u^{(0)}(t) - f\|_X \\ &\leq CT \sup_{t \in [0, T]} \|N(u^{(n)}(t))\|_X + \frac{\delta}{2} \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

Which shows that T is bounded. So the iteration (18) is Lipchitz and contraction if $CTK_\delta < 1$. Then by Banach fixed point theorem, the integral equation (17) has the unique solution $u(t)$.

Theorem 2.3.2 (Convergence theorem). Suppose the above assumption (1) is satisfied and the unique solution of equation (17) is maximal existence time and $u_n(t)$ is defined. Then there exists a $\beta \in [0, T]$ such that the partial sum $u_n(t) = \sum_{k=0}^n u_k(t)$ converges to the solution $u(t)$.

Proof: As from the above uniqueness theorem for any $\delta > 0$

$$\sup_{t \in [0, t_0]} \|u_0(t) - f\|_X \leq \frac{\delta}{2}$$

Suppose $\delta < 2a$ then by Cauchy estimates

$$\begin{aligned} &\frac{1}{k!} \|\partial_u^k N(u_0)\|_X \\ &\leq \sum_{m \geq k} \frac{m(m-1) \dots (m-k+1)}{m! k!} \|\partial_u^m N(f)\|_X \\ &\leq b \sum_{m \geq k} \frac{m(m-1) \dots (m-k+1)}{m! k!} \frac{\|u_0 - f\|_X^{m-k}}{a^m} \\ &= \frac{1}{k!} \partial_\rho^k g(\rho) \text{ where } g(\rho) = \frac{ab}{a-\rho}, \rho \\ &= \|u_0 - f\|_X \end{aligned}$$

Now by using semigroup property $\|E(t)f\|_X \leq C\|f\|_X \quad \forall f \in X, \forall t \in \mathbb{R}_+$ and the Cauchy estimates

$$\begin{aligned} \|u_1(t)\|_X &\leq C \int_0^t g(\rho(x)) dx \leq Ct g(\rho(t)) \\ &= Ct \rho'(t) \end{aligned}$$

$$\begin{aligned} \|u_2(t)\|_X &\leq C^2 \int_0^t g'(\rho(x)) g(\rho(x)) x dx \\ &= \frac{C^2 t^2}{2} \rho''(t) \end{aligned}$$

Or in general

$$\|u_k(t)\|_X \leq \frac{C^k t^k}{k!} \partial_t^k \rho(t), \quad k \in \{1, 2, 3, 4, \dots, n\}$$

Now suppose $k = n + 1$; then

$$\|u_{n+1}(t)\|_X \leq \frac{C^{n+1} t^{n+1}}{(n+1)!} \partial_t^{n+1} \rho(t) \quad (19)$$

As $\rho(t)$ is analytic so for any $\varepsilon > 0$

$$\begin{aligned} \|U_n^\varepsilon(t)\|_X &\leq \sum_{k=0}^n \varepsilon^k \|u_k(t)\|_X \\ &= \rho((1 + \varepsilon C)t) - \frac{\varepsilon^{n+1} C^{n+1}}{(n+1)!} \rho^\varepsilon(t) \end{aligned}$$

Then by the Adomian polynomial $A_n = \left(\frac{1}{n!}\right) \frac{d^n}{d\varepsilon^n} [N(\sum_{k=0}^\infty \varepsilon^k u_k)]_{\varepsilon=0}$ we get the equation (19) such that

$$\begin{aligned} \|u_{n+1}(t)\|_X &\leq \int_0^t \|E(t-x)A_n(x)\|_X dx \\ &\leq \frac{C^{n+1} t^{n+1}}{(n+1)!} \partial_t^{n+1} \rho(t) \end{aligned}$$

So the series solution $u(t) = u_0(t) + \sum_{n=1}^\infty u_n(t)$ is majorant in X then by using power series

$$\begin{aligned} \rho((1 + C)t) &= \sum_{k=0}^\infty \frac{C^k t^k}{k!} \partial_t^k \rho(t) \\ &= a - \sqrt{a^2 - 2ab(1 + C)t} \end{aligned}$$

Which converges $\forall |t| < \frac{a}{2b(1+C)}$ and by weierstrass M-test the series $u(t) = u_0(t) + \sum_{n=1}^{\infty} u_n(t)$ also converges.

2.4 Order of Convergence of ADM

To find the order of convergence we define some definitions and theorems

Definition 2.4.1 Suppose the sequence $\{S_n\}$ converges to S and if q and C are two real positive constants such that

$$\lim_{n \rightarrow \infty} \left| \frac{S_{n+1} - S}{(S_n - S)^q} \right| = C$$

Then q is the order of convergence of the sequence $\{S_n\}$. To find the order of convergence of a sequence $\{S_n\}$ suppose the Taylor series amplification of $N(y_0 + S_n)$ such that

$$N(y_0 + S_n) = N(y_0 + S) + \frac{N'(y_0 + S)}{1!} (S_n - S) + \frac{N''(y_0 + S)}{2!} (S_n - S)^2 + \dots + \frac{N^{(k)}(y_0 + S)}{k!} (S_n - S)^k + \dots$$

And as for functional equations, we have $S_{n+1} = N(y_0 + S_n)$ and $S = N(y_0 + S)$. so

$$S_{n+1} - S = N(y_0 + S) + \frac{N'(y_0 + S)}{1!} (S_n - S) + \frac{N''(y_0 + S)}{2!} (S_n - S)^2 + \dots + \frac{N^{(k)}(y_0 + S)}{k!} (S_n - S)^k + \dots \dots \dots \quad (20)$$

Now to state the order theorem [56].

Theorem 2.4.2 Let $N \in C^q[a, b]$ and if $N^k(y_0 + S) = 0$ and $N^q(y_0 + S) \neq 0$ then the sequence $\{S_n\}$ has the order q .

Proof: From the equation (20) of the above theorem

$$S_{n+1} - S = \frac{N^{(q)}(y_0 + S)}{q!} (S_n - S)^q + \frac{N^{(q+1)}(y_0 + S)}{(q+1)!} (S_n - S)^{q+1} + \dots$$

$$\frac{S_{n+1} - S}{(S_n - S)^q} = \frac{N^{(q)}(y_0 + S)}{q!} + \frac{N^{(q+1)}(y_0 + S)}{(q+1)!} (S_n - S) + \dots$$

When $n \rightarrow \infty$ then q will be the order of $\{S_n\}$. Which completes this theorem [56].

2.5 New ideas for the convergence and uniqueness of ADM

The rapid rate of convergence and accuracy of ADM is discussed in [15], [21], [30], [23]. And By using the fixed point theorem a proof of convergence is contended in [15]. New conditions to obtained the convergence are detailed in [89]. Consider a deterministic equation such that $F(y(x)) = g(x)$. Where F is the operator, a combination of linear and non-linear functions. By exploiting the Green's function then

$$Ly + Ry + Ny = g \quad (21)$$

Where L is a linear operator and R is the remainder of the linear operator. And Ny is the non-linear operator. The series solution of y by Adomian decomposition is

$$y = \sum_{n=0}^{\infty} y_n \quad (22)$$

And the nonlinear term can disintegrate as

$$Ny = f(y) \sum_{n=0}^{\infty} A_n(y_0, y_1, y_2, \dots, y_n)$$

Where A_n 's are the Adomian polynomials founded in [11], [25]. By putting the Adomian series and polynomial into the above equation and after the recurrence formula we get the solution such that

$$\varphi_m = \sum_{i=0}^{m-1} y_i, \quad \lim_{m \rightarrow \infty} \varphi_m = u$$

Now we have to prove the convergence and uniqueness for this deterministic equation [90].

Theorem 2.5.1 (Uniqueness theorem). The deterministic equation $Ly + Ry + Ny = g$ has the unique solution if $0 < \beta < 1$ where $\beta = \frac{(L_1+L_2)x^k}{k!}$.

Proof: Let X be the Banach space on the interval h . we define a mapping $F: X \rightarrow X$ such that

$$F(y(x)) = \varphi(x) + L^{-1}g(x) - L^{-1}Ry(x) - L^{-1}Ny(x)$$

Suppose y and y^* be two different solutions then

$$\begin{aligned} \|Fy - Fy^*\| &= \max_{x \in h} |-L^{-1}Ry - L^{-1}Ny + L^{-1}Ry^* + L^{-1}Ny^*| \\ &= \max_{x \in h} |-L^{-1}(Ry - Ry^*) - L^{-1}(Ny - Ny^*)| \end{aligned}$$

$$= \max_{x \in h} |L^{-1}(Ry - Ry^*) + L^{-1}(Ny - Ny^*)|$$

$$\leq \max_{x \in h} (|L^{-1}(Ry - Ry^*)| + |L^{-1}(Ny - Ny^*)|)$$

Suppose Ry is Lipchitz such that $|Ry - Ry^*| \leq L_2(y - y^*)$ where L_2 is a Lipchitz constant so

$$\|Fy - Fy^*\| \leq \max_{x \in h} (L^{-1}|Ry - Ry^*| + L^{-1}|Ny - Ny^*|)$$

$$\leq \max_{x \in h} (L_2L^{-1}|y - y^*| + L_1L^{-1}|y - y^*|)$$

$$\begin{aligned} &\leq (L_1 + L_2) \|y - y^*\| \frac{x^k}{x!} \\ &= \beta \|y - y^*\| \end{aligned}$$

As $0 < \beta < 1$, the mapping is contraction. So by fixed point theorem the equation $Ly + Ry + Ny = g$ has the unique solution.

Theorem 2.5.2 (Convergence theorem). The solution of equation $Ly + Ry + Ny = g$ will converge by ADM if $0 < \beta < 1$ and $|y_1| < \infty$.

Proof: Suppose $\{S_n\}$ be a sequence with a partial sum $S_n = \sum_{j=0}^n y_j(x)$. We had to prove that $\{S_n\}$ is a Cauchy sequence. So

$$\begin{aligned} \|S_{n+q} - S_n\| &= \max_{x \in h} |S_{n+q} - S_n| \\ &= \max_{x \in h} \left| \sum_{j=n+1}^{n+q} u_j(t) \right|, \\ q &= 1, 2, 3, \dots \end{aligned}$$

As we have that

$$N(y_0, y_1, y_2, \dots, y_n) = \sum_{j=0}^{n-1} A_j + A_n$$

And by expanding Ry to y_0

$$\begin{aligned} Ry &= \sum_{k=0}^{\infty} \frac{(y - y_0)^k}{k!} \frac{d^k}{dx^k} R(y)|_{x=x_0} \\ &= \sum_{k=0}^{\infty} \tilde{A}_k(y_0, y_1, y_2, \dots, y_k) \end{aligned}$$

And

$$\tilde{A}_0 = R(y_0) = R(S_0)$$

$$\tilde{A}_0 + \tilde{A}_1 = R(y_0 + y_1) = R(S_1)$$

So

$$\begin{aligned} \sum_{j=0}^{n-1} \tilde{A}_n &= R(S_n) - \tilde{A}_n \quad \text{and} \quad N(S_n) \\ &= \sum_{t=0}^{n-1} A_s + A_n \end{aligned}$$

As assumed that L and R are Lipchitz so

$$\|S_{n+q} - S_n\| = \max_{x \in h} \left| -L^{-1} \left(\sum_{j=n+1}^{n+q} Ru_{j-1} \right) - L^{-1} \left(\sum_{j=n+1}^{n+q} A_{j-1} \right) \right|$$

$$= \max_{x \in h} \left| -L^{-1} \left(\sum_{j=n}^{n+q-1} Ru_j \right) - L^{-1} \left(\sum_{j=n}^{n+q-1} A_j \right) \right|$$

$$= \max_{x \in h} |L^{-1}(R(S_{n+q-1}) - R(S_{n-1})) + L^{-1}(N(S_{n+q-1}) - N(S_{n-1}))|$$

$$\leq \max_{x \in h} |L^{-1}(R(S_{n+q-1}) - R(S_{n-1}))| + \max_{x \in h} |L^{-1}(N(S_{n+q-1}) - N(S_{n-1}))|$$

$$\leq \max_{x \in h} L^{-1}(|R(S_{n+q-1}) - R(S_{n-1})|) + \max_{x \in h} L^{-1}(|N(S_{n+q-1}) - N(S_{n-1})|)$$

By using the Lipchitz continuity

$$\leq L_2 \max_{x \in h} L^{-1} |S_{n+q-1} - S_{n-1}| + L_1 \max_{x \in h} L^{-1} |S_{n+q-1} - S_{n-1}|$$

$$= (L_1 + L_2) \frac{x^k}{k!} \|S_{n+q-1} - S_{n-1}\|$$

$$\text{Hence } \|S_{n+q} - S_n\| \leq \beta \|S_{n+q-1} - S_{n-1}\| \leq \beta^2 \|S_{n+q-2} - S_{n-2}\| \leq \dots \beta^n \|S_q - S_0\|$$

Suppose for $q = 1$

$$\|S_{n+1} - S_n\| \leq \beta^n \|S_1 - S_0\| \leq \beta^n \|y_1\|$$

$$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \dots + \|S_n - S_{n-1}\|$$

$$\leq (\beta^m + \beta^{m+1} + \dots + \beta^{n-1}) \|y_1\|$$

Now as $0 < \beta < 1$ so $\|S_n - S_m\| \leq \frac{\beta^m}{1-\beta} \|y_1\|$. And as in conditions $|y_1| < \infty$ so when $n \rightarrow \infty$ then $\|S_n - S_m\| \rightarrow 0$. So the sequence $\{S_n\}$ is a Cauchy sequence in X which completes this theorem.

2.6 A new modification for the Adomian polynomials.

ADM applied to a basic differential equation.

Consider a non-linear differential equation

$$\frac{d^k}{dx^k} u(x) + \alpha(x)f(u) = g(x) \tag{23}$$

Having initial conditions

$$u(0) = c_0, \quad \frac{du(0)}{dx} = c_1$$

$$\text{And } \frac{d^{k-1}u(0)}{dx^{k-1}} = c_{n-1} \tag{24}$$

When we apply the basic ADM method to the above differential equation with Adomian polynomials having the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f \left(\sum_{j=0}^{\infty} \lambda^j u_j \right) \right]_{\lambda=0} \tag{25}$$

The by applying this formula on the standard differential equation and using the conditions we get the simple arranged form of Adomian polynomials[91] such that

$$A_0 = f(u_0)$$

$$A_1 = u_1 f^{(1)}(u_0)$$

$$A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!} u_1^2 f^{(2)}(u_0)$$

$$A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0)$$

And up to so on

A new system of Adomian polynomials

The new procedure for the Adomian polynomials is abject on the new arrangements [57].

$$\bar{A}_0 = f(u_0)$$

$$\bar{A}_1 = u_1 f^{(1)}(u_0) + \frac{1}{2!} u_1^2 f^{(2)}(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0) + \dots$$

$$\bar{A}_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!} (u_2^2 + 2u_1 u_2) f^{(2)}(u_0) + \frac{1}{3!} (3u_1^2 u_2 + 3u_1 u_2^2 + u_2^3) f^{(3)}(u_0) + \dots$$

$$\bar{A}_3 = u_3 f^{(1)}(u_0) + \frac{1}{2!} (u_3^2 + 2u_1 u_3 + 2u_2 u_3) f^{(2)}(u_0) + \frac{1}{3!} (u_3^3 + 3u_3^2 (u_1 + u_2) + 3u_3 (u_1 + u_2)^2) f^{(3)}(u_0) + \dots$$

Define the partial sum $S_n = \sum_{j=0}^n u_j(x)$ then again by rearranging the above formula

$$\bar{A}_0 = f(u_0) = f(S_0)$$

$$\bar{A}_0 + \bar{A}_1 = f(u_0 + u_1) = f(S_1)$$

$$\bar{A}_0 + \bar{A}_1 + \bar{A}_2 = f(u_0 + u_1 + u_2) = f(S_2)$$

$$\bar{A}_0 + \bar{A}_1 + \bar{A}_2 + \bar{A}_3 = f(u_0 + u_1 + u_2 + u_3) = f(S_3)$$

And similarly

$$\sum_{j=0}^n \bar{A}_j(u_0, u_1, u_2, u_3, \dots, u_j) = f(S_n)$$

Or in general form

$$\bar{A}_n = f(S_n) - \sum_{j=0}^{n-1} \bar{A}_j \tag{26}$$

Which is the new principle for Adomian polynomials. For example, if $f(u) = u^4$ then the Adomian polynomials can be computed by the formula (25) and (26).

By using formula (25)

$$A_0 = f(u_0) = u_0^4$$

$$A_1 = u_1 f^{(1)}(u_0) = 4u_0^3 u_1$$

$$A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!} u_1^2 f^{(2)}(u_0) = 4u_0^3 u_2 + 6u_0^2 u_1^2$$

$$A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0) = 4u_0^3 u_3 + 12u_0^2 u_1 u_2 + 4u_0 u_1^3$$

By using formula (26)

$$\bar{A}_0 = f(u_0) = u_0^4$$

$$\bar{A}_1 = 4u_0^3 u_1 + 6u_0^2 u_1^2 + 4u_0 u_1^3$$

$$\bar{A}_2 = 4u_0^3 u_2 + 6u_0^2 (u_2^2 + 2u_1 u_2) + 4u_0 (3u_1^2 u_2 + 3u_1 u_2^2 + u_2^3)$$

$$\bar{A}_3 = 4u_0^3 u_3 + 6u_0^2 (u_3^2 + 2u_1 u_3 + 2u_2 u_3) + 4u_0 (u_3^3 + 3u_3^2 (u_1 + u_2) + 3u_3 (u_1 + u_2)^2)$$

From the above result, we concluded that $f(u)$ converges more rapidly for formula (26).

Example:

Suppose a non-linear differential equation

$$\frac{d^2 u}{dx^2} + e^{-2x} u^3 = 2e^x$$

With initial conditions

$$u(0) = 1, \quad u'(0) = 1$$

And the exact solution is $u(x) = e^x$. By using the MATHEMATICA and MATLAB solve the above differential equation and concluded that the solution converges more rapidly by formula (26) than the formula (25). The comparison between Relative Absolute Error (RAE) by new and old Adomians polynomials formula is shown in this table [57].

Table 1

The comparison between Relative Absolute Error of old and new Adomian polynomial formula

x	Exact Solution	RAE for old A_n	RAE for new A_n
0.1	1.10517	8.1814*10 ⁻¹⁴	2.5973*10 ⁻¹⁶
0.2	1.22140	6.6917*10 ⁻¹²	7.0793*10 ⁻¹⁵
0.3	1.34985	1.7117*10 ⁻¹⁰	5.1451*10 ⁻¹⁴
0.4	1.49182	9.8823*10 ⁻⁸	9.8192*10 ⁻¹³
0.5	1.64872	4.6017*10 ⁻⁶	1.0101*10 ⁻¹¹
0.6	1.82211	0.000019527	9.9109*10 ⁻⁹
0.7	2.01375	0.000087712	1.0621*10 ⁻⁸
0.8	2.22554	0.00021094	6.6317*10 ⁻⁷
0.9	2.456903	0.00078510	3.6917*10 ⁻⁶
1.0	2.718281	0.0013165	0.0000110158

2.7 Applications of the decomposition method.

2.7.1 The solution of the Heat equation by the ADM method

The main ascendancy of this method is the applicability in many fields with wide-ranging like physics, chemistry, and engineering [92], [56]. This method is highly applicable for contrastive areas such as non-linear optics, chaos theory, fermentation theory, and heat transfer [93]. The solution of the heat equation with non-linearity power is one from the applications of ADM by using this method [94]. Suppose a general form of the heat equation

$$u_y(x, y) = u_{xx} + u^m \tag{27}$$

Where u^m is the power of non-linearity and its initial conditions will be

$$u(x, 0) = f(x)$$

Suppose w is an operator of an approximation such that

$$w_y u(x, y) = w_{xx} u + u^m \tag{28}$$

And suppose its inverse operator exists such that $w_y^{-1}(\cdot) = \int_0^y \cdot dy$. so by employing the inverse operator on equation (28)

$$w_y^{-1} w_y u(x, y) = w_y^{-1} w_{xx} u + \epsilon w_y^{-1} u^m$$

$$u(x, y) = u(x, y) + w_y^{-1} w_{xx} u + \epsilon w_y^{-1} u^m \tag{29}$$

And the series solution of $u(x, y)$ by Adomian decomposition is

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \tag{30}$$

And Adomian polynomials will be

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \right]_{\lambda=0}$$

So

$$N u(x, y) = \sum_{n=0}^{\infty} A_n \tag{31}$$

By putting equation (30), (31) into (29)

$$u(x, y) = u(x, 0) + w_y^{-1} w_{xx} \sum_{n=0}^{\infty} u_n(x, y) + w_y^{-1} \sum_{n=0}^{\infty} A_n$$

Then by the recurrence relation

$$u_0(x, y) = f(x)$$

$$u_{n+1}(x, y) = w_y^{-1} w_{xx} u_n(x, y) + \epsilon w_y^{-1} A_n \quad ,$$

Where $n = 0, 1, 2, 3, \dots$

So

$$u_1(x, y) = w_y^{-1} w_{xx} u_0 + w_y^{-1} A_0$$

$$u_2(x, y) = w_y^{-1} w_{xx} u_1 + w_y^{-1} A_1$$

$$u_3(x, y) = w_y^{-1} w_{xx} u_2 + w_y^{-1} A_2$$

$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$

$$u_n(x, y) = w_y^{-1} w_{xx} u_{n-1} + w_y^{-1} A_{n-1}$$

Now suppose that ψ_y is its approximate solution then

$$\psi_y = \sum_{n=0}^{\infty} u_n(x, y) \tag{32}$$

From equation (31) and (32) we consummated that

$$u(x, y) = \lim_{y \rightarrow \infty} \psi_y(x, y)$$

The connotation of the approximate or exact solution is disclosed by the adaptability of these problems [95], [96], [97]. Pumak bestows the ADM method to treasure the exact solution of the heat equation with linear and non-linear powers[98].

Example 1.

To illustrate more this topic we suppose a non-linear heat equation such that

$$u_y(x, y) = u_{xx} - 2u^3 \tag{33}$$

Which is the non-linear heat equation with a non-linear power u^3 with condition

$$u(x, 0) = \frac{1 + 2x}{x^2 + x + 1}$$

And its exact solution is

$$u(x, y) = \frac{1 + 2x}{x^2 + x + 6y + 1} \tag{34}$$

So to solve this equation using again approximation operator w such that

$$u(x, y) = u(x, 0) + w_y^{-1} w_{xx} u - 2w_y^{-1} u^3$$

And the Adomian polynomials will be as

$$A_0 = -2u_0^3$$

$$A_1 = -6u_0^2 u_1$$

$$A_2 = -6(u_0 u_1^2 + u_0^2 u_2)$$

Hence

$$u_0 = \frac{1 + 2x}{x^2 + x + 1}$$

$$u_1 = w_y^{-1} w_{xx}(u_0) - 2w_y^{-1}(w_0^3) = \frac{-6(1 + 2x)}{x^2 + x + 1} y$$

$$u_2 = w_y^{-1} w_{xx}(u_1) - 6w_y^{-1}(u_0^2 u_1) = \frac{36(1 + 2x)}{(x^2 + x + 1)^3} y^2$$

$$u_3 = w_y^{-1} w_{xx}(u_2) - 6w_y^{-1}(u_0^2 u_2 + u_1^2 u_0) = \frac{-216(1 + 2x)}{(x^2 + x + 1)^4} y^3$$

And up to so on. so the solution of $u(x, y)$ by decomposition series

$$u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots$$

$$u(x, y) = \frac{1 + 2x}{x^2 + x + 1} - \frac{6(1 + 2x)}{x^2 + x + 1} y + \frac{36(1 + 2x)}{(x^2 + x + 1)^3} y^2 - \frac{216(1 + 2x)}{(x^2 + x + 1)^4} y^3 + \dots$$

By simplifying and terminating the infinite terms, the approximate solution of $u(x, y)$ will be

$$u(x, y) \approx \frac{1 + 2x}{x^2 + x + 6y + 1}$$

Which is the solution of the above heat equation by decomposition method [98].

2.7.2 | The solution of the Wave Equation by using ADM

The wave equations are non-trivial PDE's that occurs at different places. The most common physical phenomena of wave equations wave propagation that occurs in daily life [99]. Consider a standard wave equation for the basic implementation of the decomposition method

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \tag{35}$$

Where $u = \alpha(x - ct)$ shows the wave that conciliates the wave equation (35), "c" is any constant and in simple form equation (35) can be written as

$$u_{xx} = c^{-2} u_{tt}$$

Which is the linear 2nd order homogeneous wave equation with initial conditions

$$u(0, t) = f(t) \quad , \quad u_x(0, t) = h(t)$$

Where equation (35) may have the solution in the form of

$$u(x, t) = \alpha(x - ct) + \beta(x + ct) \tag{36}$$

And equation (35) can be written in differential operator "I" such that

$$I_t(u(x, t)) = I_x(c^2 u(x, t)) \tag{37}$$

Where "I" is the differential operator i.e. $I_x = \frac{\partial^2}{\partial x^2}$, $I_t = \frac{\partial^2}{\partial t^2}$. suppose the inverse of the differential operator I_t^{-1} exists and defined as

$$I_t^{-1} = \int_0^t \int_0^t (.) dr dr$$

Then equation (37) becomes

$$I_t^{-1}[I_t(u(x, t))] = c^2 I_t^{-1}[I_x(u(x, t))] \quad (38)$$

Now by using the initial conditions then we get

$$u(x, t) = g(t) + xh(t) + c^2 I_t^{-1}[I_x(u(x, t))] \quad (39)$$

And $u(x, t)$ can be decomposed in series form by Adomian series such that

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

So equation (39) will become

$$\sum_{n=0}^{\infty} u_n(x, t) = g(t) + xh(t) + c^2 I_t^{-1} \left[I_x \left(\sum_{n=0}^{\infty} u_n(x, t) \right) \right] \quad (40)$$

$$u_0 = g(t) + xh(t)$$

$$u_1 = c^2 I_t^{-1} [I_x(u_0)]$$

$$u_2 = c^2 I_t^{-1} [I_x(u_1)]$$

$$u_3 = c^2 I_t^{-1} [I_x(u_2)]$$

⋮

$$u_{n+1} = c^2 I_t^{-1} [I_x(u_n)]$$

Then the exact solution may be found by approximation [100], [99]

$$u(x, t) = \lim_{n \rightarrow \infty} \alpha_n$$

Example 2.

Suppose a wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < \pi$$

With initial conditions

$$u(x, 0) = \sin 3x, \quad u_t(x, 0) = 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0$$

To solve this problem first we use the D'Alemberts formula which can be defined as

$$u(x, t) = \frac{1}{2} [g(x - ct) + g(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(r) dr$$

And as in above problem $h(t) = 0, a = 1$, $g(x) = \sin 3x$ so

$$u(x, t) = \frac{1}{2} [\sin 3(x - t) + \sin 3(x + t)] + 0$$

By using the identity

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta, \quad \sin(\alpha - \beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \end{aligned}$$

$$u(x, t) = \frac{1}{2} [\sin 3x \cos 3t - \cos 3x \sin 3t + \sin 3x \cos 3t + \cos 3x \sin 3t]$$

$$u(x, t) = \sin 3x \cos 3t$$

Now by using the ADM

$$u_0 = \sin 3x$$

$$u_1 = I_t^{-1} [I_x(\sin 3x)] = \frac{-9}{2} t^2 \sin 3x$$

$$u_2 = I_t^{-1} \left[I_x \left(\frac{-9}{2} t^2 \sin 3x \right) \right] = \frac{81}{24} t^4 \sin 3x$$

$$u_2 = I_t^{-1} \left[I_x \left(\frac{81}{24} t^4 \sin 3x \right) \right] = \frac{-729}{720} t^6 \sin 3x$$

And up to so on. So the comparative solution of the above wave equation

$$u = u_0 + u_1 + u_2 + u_3 + \dots$$

$$u = \sin 3x - \frac{9}{2} t^2 \sin 3x + \frac{81}{24} t^4 \sin 3x - \frac{729}{720} t^6 \sin 3x + \dots$$

$$u = \sin 3x - \frac{(3t)^2}{2!} \sin 3x + \frac{(3t)^4}{4!} \sin 3x - \frac{(3t)^6}{6!} \sin 3x + \dots$$

$$u = \sin 3x \left[1 - \frac{(3t)^2}{2!} + \frac{(3t)^4}{4!} - \frac{(3t)^6}{6!} + \dots \right]$$

$$u(x, t) = \sin 3x \cos 3t$$

Which is the exact solution of the given wave equation by using ADM[100].

4 Conclusions

The convergence of ADM has been proved for different linear and non-linear equations. From the above results, we consummated that this method is beyond compare method to solve linear and non-linear equations. This method is applicable, upstanding, and without a restrictive assumptions method. This method can use directly without linearizing any linear or non-linear problem therefore this method gives a more reliable, accurate, and rapidly convergent solution. Numerical examples show that this method is more capable, more decent, more adaptable, and more spark to both solutions approximate and exact for heat and wave equations.

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